Collusion in an investment game

Thomas Fagart

Université de Paris 1 Panthéon Sorbonne, UFR 02 Sciences Economiques.
Mémoire du master 2 recherche Economie Théorique et Empirique

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Dirigé par Pierre Fleckinger

Soutenu par Thomas Fagart

Introduction

If collusion was often considered in a market facing uncertainty (Bagwell and Staiger (1997)), or imperfect information (Athey and Bagwell (2008), Harrington and Skrzypacz (2010)), the relationship between collusion and investment is less known. That is the purpose of what follows.

This work studies a dynamic game in discrete time with infinite periods. In each period firms make two decisions, investment (or disinvestment) in production capacity and the quantities they produce. Companies can choose to increase or reduce capacity. The irreversibility of decisions is modeled by the difference between purchase price and sale of building (when the gap is zero, the decisions are totally reversible). In each period firms are competing in Cournot, the quantities produced are of course limited by production capacity. The model is presented in section 1.3.

In comparison with the Account, Jenny and Rey (2003), capacity is endogenous, modified in each period, and the game of competition is a game of Cournot competition while Account Jenny and Rey (2003) are interested in a game competition in Bertrand-Edgeworth. In comparison with Boyer, Lasserre and Moreaux (2010), demand is not random, so there is no uncertainty and the equilibrium concept used is far less restrictive than the Markov equilibrium. Production capacities are not discrete and are not irreversible. These papers are presented in section 1.1 and 1.2.

To define the collusion, it is necessary to determine a non-collusive equilibrium. In a repeated game, this benchmark equilibrium is constituted by the repetition of the equilibrium of the one shot game. In a stochastic game, as here, we can not implement this solution. We must therefore define a reference equilibrium. This point is develloped in section 2.1. In section 2.2 and 2.3 we prove the existence and the unicity of this benchmark equilibrium.

If we discretize the game (ie the actions of the players belong to a finite space, which can be chosen infinitesimally large), section 3.1 presents a folk theorem (the proof uses the result of Horner, Sugaya , Old and Takahashi (2010)). This theorem tells us that when the discount rate tends to 1, the set of equilibrium payoff vectors tends to the set of equilibrium payoff vectors of the infinitely repeated Cournot game (without cost or production capacity). The theorem is therefore a borderline result, which gives an equivalence between this game and the Cournot game (infinitely repeated) when players are infinitely patient. Finally, section 3.2 studies conditions for the existence of a specific collusive equilibrium (the Grim-Trigger equilibrium in capacities).

1 Problematic

Impact of capacity of production on the collusion

To study the collusion, a dynamic model of infinitely repeated game in which enterprises compete at each period is used. Stage competitive models used are Cournot, Bertrand, or variations of these games. The first articles implicitly assumed that each company can meet any demand, and this at each period of the game The idea that companies cannot serve the whole market because they are limited in production capacity, and the impact due to this restriction on collusion, have been recently studied in an article Compte, Jenny and Rey (2003).

In this article, companies have an exogenous production capacity (which does not vary over time) and compete an infinite number of times in Bertrand-Edgeworth framework (with inelastic demand) (ie sales are limited by their capacity). (The time is modeled discretely). The authors determine the level of the minimum discounted rate allowing collusion (defined as allowing companies to make profits above the static equilibrium of the game).

This threshold depends on the initial capacities given *a priori*. More precisely, if we note k^i the capacity of the firm and M the mass of consumers, two different cases exist:

- If all firms except the dominant (which has the largest capacity) can cover the whole market: There is collusion if the discount rate δ is larger than $1 - \frac{M}{K}$ (where K is the sum of capacities of all firms on the market). The idea is that smaller capacities increase collusion because they reduce the incentives to deviate. Note that this result corresponds to the classical results when firms are not limited ($\delta > 1 - \frac{1}{n}$, where n is the number of firms).

- In the other cases (where "small" firms cannot cover the whole market) the discount rates allowing collusion are greater than $\frac{k^n}{K}$ (where k^n is the capacity of the largest firm). A reduced capacity for small firms therefore reduces the possibility of collusion by reducing the opportunities to punish collusion.

The production capacity thus affect the collusion that may appear in game equilibria. The impact of capacity depends on a degree of asymmetry in capacities and greater capacities for small firms can encourage collusion.

Recall that in this article, the authors assume that the production capacity is exogenously fixed throughout the game. In other words the question of choosing the optimal investment for a company is not asked.

There exists another article, written by Feuerstein and Gersbach (2003), which study collusion (more exactly the existance of a punishment strategy, the Grim-Trigger one) when investments are irreversible. They find that irreversibility increase the possibilities of collusion. However, the investment is costless, which means that all firms can increase their capacities freely; even if the capacity has a per period cost. Hence it is more a problem of commitment than a problem of optimal investment.

Optimal investment choices

There exists a range of literature which concerns the optimal choice of capacity of production, the theory of real options. It has mainly focused on the study of investment choices in an uncertain environment where the future profits of the investment in question does not depend on the choices of other firms in the market (see the book by Dixit and Pindyck (1994)). The study of the relationship between competition and investment is more recent, but does not take into account the mechanisms of collusion. It is summarized for example in Knight-Roignant, Flath, Huchzermeier and Trigeorgis (2010). Competition pushes firms to invest earlier. Some characteristics (the number of firms on the market, the greater divisibility of investments or the existence of an advantage to invest first) encourage firms to invest even earlier. Other characteristics (such as firm heterogeneity, the existence of an advantage to invest in second or in some cases incomplete information), encourage them instead to invest later.

Articles by Boyer, Lasserre, Mariotti and Moreaux (2004) and Boyer and Moreaux Lasserre (2010), study the interaction between competition and investment choices. These are the only ones discussing some collusion in capacity.

The first article examines the case of a duopoly facing a Bertrand-Edgeworth competition in continuous time with infinite horizon. Firms start with a production capacity equal to zero and decide when increasing their capacity (so the first decision to increase its capacity is in fact a decision of entry). These investments are irreversible and costly. Capacities are assumed discrete (each firm can not hold that 0, 1 or 2 unit) and the market is covered by two units of capacity. The reserve price of the game to the Bertrand-Edgeworth is assumed to be random. Specifically it is a Brownian diffusion:

$$dP_t = \alpha P_t dt + \sigma P_t dW_t$$

where σ is the volatility of prices, the growth rate α and r the interest rate.

The equilibrium concept used is Markov perfect equilibrium (where owned capacities are the state space) and subgame perfect Nash equilibrium. This implies that the strategies, capabilities and price on a given date, only depend on the installed capacity to the previous date. Punishment in period t can only depend on the installed capacity in time $t-1$ and not on the past history of the game. Authors speak of collusion when firms agree to delay their investments. Equilibria depend on the growth rate and on the volatility of the reserve price.

If $r < 2\alpha + \sigma^2$ (high volatility or rate of growth exceeds the rate of interest), the Markov equilibrium is unique, and the two companies buy each a capacity equal to 1, and realize this investment when a monopoly would have invested. There is therefore no preemption by competitors (competition does not encourage them to invest earlier), nor reduction of profits in the industry: everything happens as if companies maximize the joint profit.

If $r > 2\alpha + \sigma^2$ (low volatility and average growth rate below the interest rates) there are several Markov equilibria. When the initial price is low enough, whatever the equilibrium is, the two companies invest at the same time a capacity equal to 1, but do it earlier than would do a monopoly. More precisely, each firm invests at the time when, if it did not invest, the opponent would buy two units of capacity. When the initial price is high, different equilibria give different ways to invest. In particular, there exists an equilibrium where each firm buys two capacities (thereby inducing a zero profit).

Boyer, Lasserre and Moreaux (2010) generalize these results to the case of Cournot competition. The duopoly compete *à la Cournot* in continuous time (but quantities are limited for each date by production capacity). Capacities are discrete (the market is covered by a number N of units of capacity and a firm may have a number 1,2, .., N capacity), initial capacities are not necessarily zero, and companies may increase their investment irreversibly. The price is assumed to be random and we have:

$$P_t = Y_t D^{-1}(q_t^1 + q_t^2),$$

where Y_t follows a diffusion browienne and where q_t^1 and q_t^2 are the quantities produced by firms 1 and 2. Such specification of the stochastic demand implies that the quantities produced (given capacities) do not depend on the random variable Y_t (production costs are assumed zero). Another way to see this hypothesis is to consider that the price of investment is stochastic and the demand is not.

As in their previous paper, the authors characterize a tacit collusion equilibrium as a period during which firms invest simultaneously, by contradiction with the other type of equilibrium, said preemption, in which firms invest before their opponents.

The authors find that competition may be tougher in the early stages after developement market is. Indeed, when one firm or less is on the market, the collusive equilibrium cannot be realized. This implies that the initial investment comes sooner than what is socially optimal.

When the initial capacities are different, the smaller firm takes the risk of investing, and eventually catches up the largest firms (although it tries to maintain a capacity gap).

Periods of collusion may happen in equilibrium when the two firms are in the market. As in their article of (2004), increasing the volatility or the market growth rate makes these episodes of tacit collusion more likely. When firms have equal sizes, such strategies maximize joint profits, which is not the case when firms have different sizes.

Model and notation

In our model, n firms produce an homogenous good and compete on several periods. The time is discrete and the horizon is infinite. At each time t, the profit of firm i depends on its quantity of good produced q_t^i, and on the total quantities produced by all the other firms, q_t^{-i}. A firm cannot choose a quantity bigger than the one induced by its capacity of production at date t, K_t^i. Production costs are normalized to zero (without loss of generality when the demand is linear). The profit of time t due to the choice of sold quantities is:

$$\tilde{\pi}_t^i(K_t^i, q_t^i, q_t^{-i}) = D^{-1}(q_t^i + q_t^{-i}) \min(q_t^i; K_t^i).$$

The capacity of production of the firm is assumed to be quasi-irreversible, wich means than there exists a buying price p^+ which is strictly superior to a selling price p^-. The case of total irreversibilty can be treated by assuming $p^- = 0$, if firms can decrease their capacities without cost, or by assuming that $p^- = -\infty$. The total profit of firm i at date t is

$$\pi_t^i(K_t^i, q_t^i, K_t^{-i}, q_t^{-i}) = \tilde{\pi}_t^i(K_t^i, q_t^i, q_t^{-i}) - p^+(K_t^i - K_{t-1}^i)^+ + p^-(K_t^i - K_{t-1}^i)^-.$$

We assume that each firme has the same discount factor, so the intertemporal profit of firm i is:

$$\Pi^i(K_t^i, q_t^i, K_t^{-i}, q_t^{-i}) = (1-\delta) \sum_{t=1}^{+\infty} \delta^{t-1} \pi_t^i(K_t^i, q_t^i, K_t^{-i}, q_t^{-i})$$

Where K_0^i is the initial capacity of firm f.

Note that this is a dynamic game but it is not a repeated game, because the pay-off of one period depends on existing capacities in the previous period. The classical results (as the folk theorem) on repeated games cannot be applied.

We also assume that the inverse demand function is linear,

$$D^{-1}(Q_t) = A - Q_t.$$

Hence, we assume that the firms quantitites and the capacities cannot be greater than A, as A is sufficient to cover the market. So firms can choose their capacities and quantities in $[0, A]$. We note this game $GI([0, A])$.

2 Non-collusive equilibrium

Definition

The definition of the notion of collusion is not always an easy issue. Usually, when we face a repeated game with only one Nash equilibrium in the stage game, the collusion is defined as a sub-game perfect equilibrium (of the dynamic game) which gives higher profits that the repetition of the stage game Nash equilibrium. This repetition of the stage game Nash equilibrium is a sub-game perfect equilibrium of the repeated game as we are indeed in a repeated game. However, in our case, we are not facing a repeated game. So we need to find a sub-game perfect equilibrium (which will be the benchmark case) to define collusion.

The first idea is to restrict the class of equilibria to markov perfect equilibria. Indeed, under non-collusive competition, the firms should take their decisons only according to the level of capacities of the industry, and not according to the history. In this case, firms choose at each period, the quantity of the one-shot Cournot game with limited capacity. However, the capacity level can depend of the capacity of the previous period, and so this restriction is not strong enought to forbid the possibility of punition. For exemple a Grim-Trigger strategy (like the one considered in sub-section 3.2), is a markovian perfect equilibrium which permits to index punishment on capacities.

Another point is to consider that, as demand and price are constant in time and there are no information issues, the quantities and capacities must be constant in time after a while. In our case, firms have no competitive reasons which justify to delay the investment compared to this "long time" investment level (investment prices are linear). So we can consider the markov perfect equilibrium in which, on the equilibrium path, firms install in the first period their long term capacity (and do not change their capacity after that). However, this is not sufficient to define the benchmark equilibrium, as the Grim-Trigger strategy previously mentionned verifies such condition.

Hence we define the competitive equilibrium as the one in which firms invest to a given level of capacity in the first time and never change it, whatever the industry capacity is at the previous period.

Definition 1 *: A competitive (or benchmark) equilibrium is defined as strategies in which firms invest at the first period, do not move their capacity after that, and produce at maximal capacity. Formaly it gives:*

$$\forall i, \ \left(K_t^i, q_t^i\right) = (q^i, q^i).$$

Note that this is obviously a markov perfect equilibrium, as it does not depend of any history. The purpose of the present part is to show that the definition above define an unique sub-game perfect equilibrium (the benchmark equilibrium) and to characterize it. Proposition 1 gives the best response of a firm facing competitors playing such strategies, and Theorem 1 etablishes the existence and the unicity of the competitive equilibrium, and characterizes it. Before presenting these results in the general case, we give the intuition when there is only two firms on the market.

Duopoly case

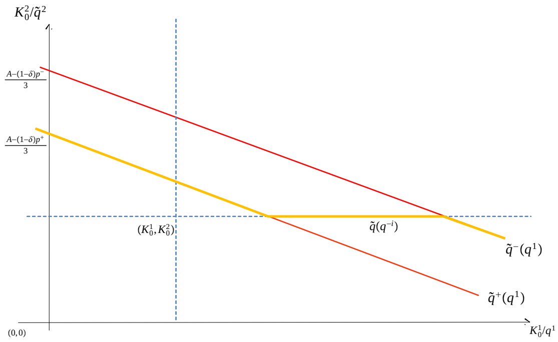
The first point is to find the best response of one firm when the other firms play the benchmark strategy. We find that the best response is also to play the benchmark strategy, with q^i (the level of capacity and of produced quantity) such that:

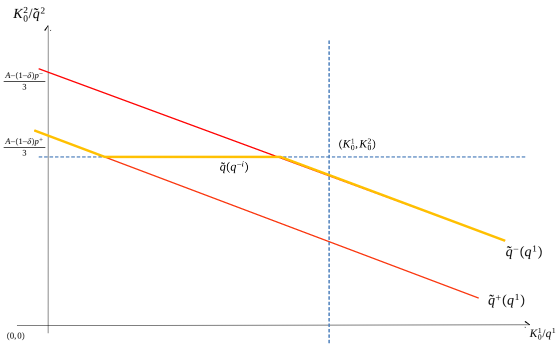
$$q^i = \left\{ \begin{array}{c} \tilde{q}^+(q^j) \text{ if } K_0^i \leq \tilde{q}^+(q^j) \\ K_0^i \text{ if } \tilde{q}^+(q^j) < K_0^i < \tilde{q}^-(q^j) \\ \tilde{q}^-(q^j) \text{ if } K_0^i \geq \tilde{q}^-(q^j) \end{array} \right\},$$

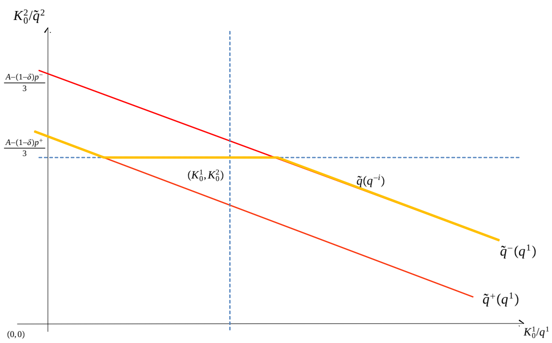
where

$$\begin{aligned}\tilde{q}^{+}(q^{j}) &= \frac{A - q^{j} - (1-\delta)p^{+}}{3}, \\ \tilde{q}^{-}(q^{j}) &= \frac{A - q^{j} - (1-\delta)p^{-}}{3}.\end{aligned}$$

Best Responses of player 2 for different values of K_0*:*

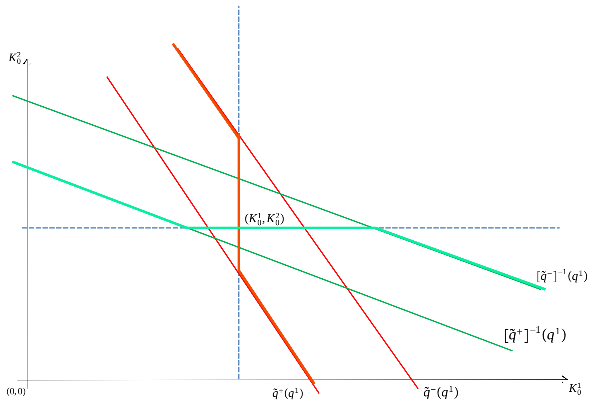






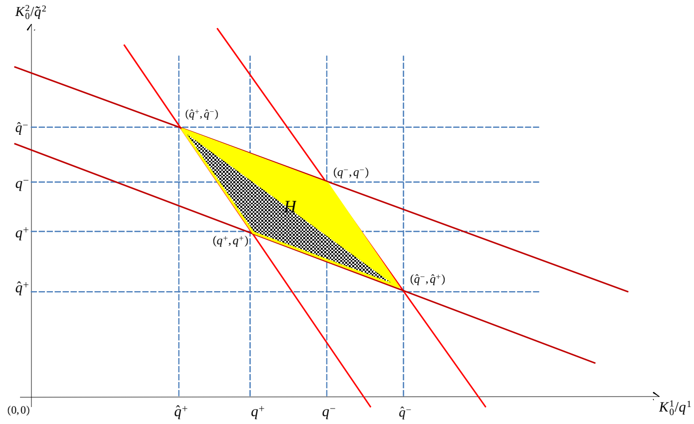
We can see that the best response is asymetric, depends on the initial capacity, and is smaller than the best response of Cournot without cost. Furthermore, there exist initial capacities such that the best response is to not change this capacities (see last graph in *Best Responses of player 2 for different values of* K_0). The next step is to see that there exist initial capacities for which neither firm 1 nor firm 2 change their capacities. Indeed, in the graph *Best responses of players 1 and 2*, the equilibrium, given by the intersection of the best responses of player 1 and 2, is (K_0^1, K_0^2).

Best responses of players 1 and 2:



This implies the existence of a space of initial capacities, H, such that for each vector of initial capacities in H, the benchmark equilibrium is to not change these capacities (and to produce at maximal capacities). This space is delimited by the straights of best response (more precisely the "partial" best response q^+ and q^-).

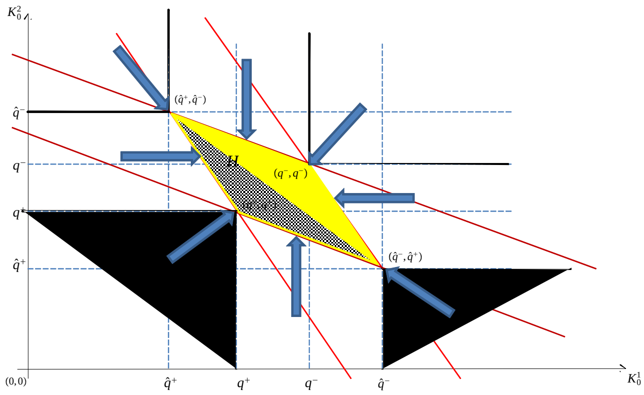
No-move space in two dimensions:



Note that we have $\hat{q}^+ = \frac{A-(1-\delta)(2p^+-p^-)}{3}$, and $\hat{q}^- = \frac{A-(1-\delta)(2p^--p^+)}{3}$. We can see that H is the convex hull of the intersection points of the straights of best responses ($\tilde{q}^+$ and $\tilde{q}^-$). The next step is to find how the firms change their capacities when they do not begin in H. In fact they will join H in one shot (invest to a vector of capacity in H), but the point of H depends on some areas on initial capacities, what is summarized in the following table:

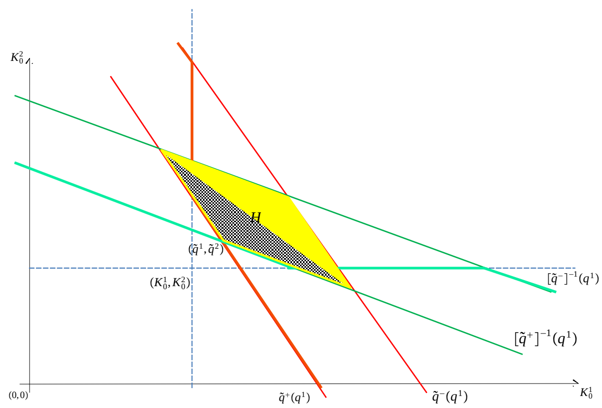
Condition on (K_0^1, K_0^2)	Strategy of player 1	Strategy of player 2
$(K_0^1, K_0^2) \in H$	$(K_t^1, q_t^1) = (K_0^1, K_0^1)$	$(K_t^2, q_t^2) = (K_0^2, K_0^2)$
$K_0^1 \leq q^+$ and $K_0^2 \leq q^+$	$(K_t^1, q_t^1) = (q^+, q^+)$	$(K_t^2, q_t^2) = (q^+, q^+)$
$K_0^1 \geq q^-$ and $K_0^2 \geq q^-$	$(K_t^1, q_t^1) = (q^-, q^-)$	$(K_t^2, q_t^2) = (q^-, q^-)$
$K_0^1 \leq \hat{q}^+$ and $K_0^2 \geq \hat{q}^-$	$(K_t^1, q_t^1) = (\hat{q}^+, \hat{q}^+)$	$(K_t^2, q_t^2) = (\hat{q}^-, \hat{q}^-)$
$K_0^1 \geq \hat{q}^-$ and $K_0^2 \leq \hat{q}^+$	$(K_t^1, q_t^1) = (\hat{q}^-, \hat{q}^-)$	$(K_t^2, q_t^2) = (\hat{q}^+, \hat{q}^+)$
$K_0^1 < \tilde{q}^+(K_0^2)$ and $K_0^2 \in]q^+, \hat{q}^-[$	$(K_t^1, q_t^1) = (\tilde{q}^+(K_0^2), \tilde{q}^+(K_0^2))$	$(K_t^2, q_t^2) = (K_0^2, K_0^2)$
$K_0^1 > \tilde{q}^-(K_0^2)$ and $K_0^2 \in]q^+, \hat{q}^-[$	$(K_t^1, q_t^1) = (\tilde{q}^-(K_0^2), \tilde{q}^-(K_0^2))$	$(K_t^2, q_t^2) = (K_0^2, K_0^2)$
$K_0^1 \in]q^+, \hat{q}^-[$ and $K_0^2 < \tilde{q}^+(K_0^1)$	$(K_t^1, q_t^1) = (K_0^1, K_0^1)$	$(K_t^2, q_t^2) = (\tilde{q}^+(K_0^1), \tilde{q}^+(K_0^1))$
$K_0^1 \in]q^+, \hat{q}^-[$ and $K_0^2 > \tilde{q}^-(K_0^1)$	$(K_t^1, q_t^1) = (K_0^1, K_0^1)$	$(K_t^2, q_t^2) = (\tilde{q}^-(K_0^1), \tilde{q}^-(K_0^1))$

Move from the different areas of initial capacities:



This result can be proved by Theorem 1, which we present in the following sub-section, or by reasoning case by case using the best response, as presented in the following graphic.

Equilibrium when initial capacities are not in H:



General case

Proposition 1 (Best response) *: For any firm i, assume that others set their capacities in period 1, and produce at maximal capacities, so $(K_t^j, q_t^j) =$*

(q^j, q^j) $\forall t$. *Then the best response of firm* i *is also to set its capacity at period 1, and to produce at maximal capacity, which means:*

$$(K_t^i, q_t^i) = (\tilde{q}(q^{-i}), \tilde{q}(q^{-i})). \tag{1}$$

The optimal level of capacity $\tilde{q}(q^{-i})$ *is given by:*

$$\tilde{q}(q^{-i}) = \left\{ \begin{array}{c} \tilde{q}^+(q^{-i}) \text{ if } K_0^i \leq \tilde{q}^+(q^{-i}) \\ K_0^i \text{ if } \tilde{q}^+(q^{-i}) < K_0^i < \tilde{q}^-(q^{-i}) \\ \tilde{q}^-(q^{-i}) \text{ if } K_0^i \geq \tilde{q}^-(q^{-i}) \end{array} \right\}, \tag{2}$$

where $\tilde{q}^+(q^{-i}) = \frac{A-q^{-i}-(1-\delta)p^+}{n+1}$, $\tilde{q}^-(q^{-i}) = \frac{A-q^{-i}-(1-\delta)p^-}{n+1}$, *(with* $q^{-i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} q^j$*).*

Proof. [Proof (Best response)] : see annex 1. ■

Note that there exists an interval of q^{-i}, where firm i stays at its initial capacity:

$$[A - (1-\delta)p^+ - (n+1)K_0^i, A - (1-\delta)p^- - (n+1)K_0^i].$$

This is due to the quasi-irreversibility of the investment. Indeed, if the investment is fully reversible (which means that $p^+ = p^-$) then this interval does not exist (formally, it is reduced to a single point, but as the best response is continuous, it does not matter).

When firms install a capacity that differ from its initial capacity, it is the best response of Cournot (in a one stage game) with some cost which differs depending on whether the firm will reduce or increase its capacity. When the firm increase its capacity, the cost, $(1-\delta)p^+$, reflects the fact that even if the firm installs its capacity at the first period, the cost is shared between all the periods. Indeed, the more patient is the firm ($\delta \to 1$) the more it will invest, even if it pays the same price, as the firm values the future more and absorbs a larger share of its investment in the future.

When the firm decreases its initial capacity, it also faces a cost, $(1-\delta)p^-$, which must be viewed as an opportunity cost. Indeed, when the firm has decreased its capacity until the Cournot level (without cost) it can do better by selling some capacities (as the derivative of the profit is close to 0 when we are near enough to the cournot level without cost).

So the best response is asymetric, depending of the initial capacity, and smaller than the best response of Cournot without cost. As the best response is to not change its initial capacity for some opponent strategies, we can define the no-move space H, as a space of capacities such that, if the vector of initial capacity of the industry is in H, firms do not change their capacity at the benchmark equilibrium.

Proposition 2 (No-move space) *: Let*

$$H = \left\{ (q^1, ..., q^n) : \forall i, q^i \in \left[\frac{A - (1-\delta)p^+ - \sum_{j \neq i} q^j}{2}, \frac{A - (1-\delta)p^+ - \sum_{j \neq i} q^j}{2} \right] \right\}, \tag{3}$$

and

$$q^+ = \frac{A - (1-\delta)p^+}{n+1}, \; q^- = \frac{A - (1-\delta)p^-}{n+1}. \tag{4}$$

Then, H contains the points $(q^-, .., q^-)$ and $(q^+, .., q^+)$

Most important, if the initial quantities are in H, the only competitive equilibrium is to not change the capacities, so:

$$\forall i \; (K_t^i, q_t^i) = (K_0^i, K_0^i).$$

Proof. [Proof (No-move space)] : It is obvious that the points $\left(\frac{A-(1-\delta)p^-}{n+1}, .., \frac{A-(1-\delta)p^-}{n+1}\right)$ and $\left(\frac{A-(1-\delta)p^+}{n+1}, .., \frac{A-(1-\delta)p^+}{n+1}\right)$ are in H.

If $(K_0^1, .., K_0^n) \in A$, then, whatever i, K_0^i is the best response to the other initial capacities by Proposition 1.

If $(K_0^1, .., K_0^n)$ is an Nash equilibrium, then for all j, K_0^j is the best response to other initial capacities, but if $(K_0^1, .., K_0^n) \notin A$, there exists j such that $2K_0^j \notin \left[A - (1-\delta)p^+ - \sum_{j \neq i} K_0^j, A - (1-\delta)p^+ - \sum_{j \neq i} K_0^j\right]$, so K_0^j is not a best response to other initial capacities, and $(K_0^1, .., K_0^n)$ is not a subgame perfect Nash equilibrium. ■

We can characterize this space as a convex hull of the intersection points of straights of best response ($q^+(.)$ and $q^-(.)$).

Geometric characterization of H:

For all k, $E \subset \{1, .., n\}$, let $a_E^k = (q_1, .., q_n)$ be the point such that:

$$\begin{aligned} \forall j &\in E, q^j = \frac{A - (1-\delta)\left[(n+1-k)p^+ - (n-k)p^-\right]}{n+1}, \\ \forall j &\notin E, q^j = \frac{A - (1-\delta)\left[(k+1)p^- - kp^+\right]}{n+1}. \end{aligned}$$

Let $\bar{H} = \{a_E^k : \exists k \in \{0, .., n\}, E \subset \{1, .., n\} \ with \ card(E) = k\}$.

Then we have $H = conv(\bar{H})$, with $conv(\bar{H})$ is the convex hull of $\bar{H}$, which means the most little convex set that contains $\bar{H}$.

Note that in some asymetric case, there can be some points of H where one firm produces more than the usual Cournot quantity (without cost). This is due to the fact that the other firms produce less.

The main theorem consists in understanding how firms join the space H when they begin in another place. More precisely, it provides the existence and the unicity of the benchmark equilibrium, characterizing it as the point of H which is the nearest of initial capacities, for some distance.

Theorem 1 (Benchmark equilibrium) *: For any initial capacities* $(K_0^1, .., K_0^n) \in \mathbb{R}_+^n$*, there exists one and only one benchmark equilibrium. Firm i installs capacity $\breve{q}^i$ at the first period and always products at full capacity. The vector of capacities $(\breve{q}^1, .., \breve{q}^n)$ is the only vector which solves* $\min_{(q^1,..,q^n)\in H} \sum_{i=1}^n |q^i - K_0^i|$.

Proof. [Proof (Benchmark equilibrium)] : See annex 2. ■

This theorem has several implications.

If $K_0 \leq (q^-, .., q^-)$, then the benchmark equilibrium is

$$\forall i, \ (K_t^i, q_t^i) = (q^-, q^-).$$

So when all the firms are small enough, the competitive equilibrium is unique and symetric.

If $K_0 \geq (q^+, .., q^+)$, then the benchmark equilibrium is

$$\forall i, \; (K_t^i, q_t^i) = (q^+, q^+),$$

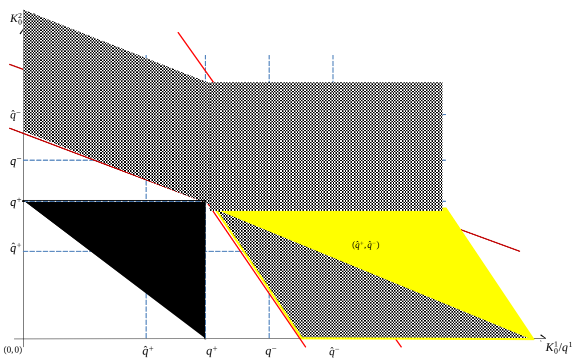
So when all the firms are big enough, the competitive equilibrium is also unique and symetric.

Of course, this competitive equilibrium is not the same in both cases. A way to interpret this result is to think that the industry faces a non-expected demand shock. In this case, for the same level of demand obtained after the shock, the industry capacities (and the thougness of the competition) are not the same if the demand was high or low before the shock.

Note that when the investment is totally reversible ($p^+ = p^-$) H is a single point, which means that initial capacities have no role in the competitive equilibirum, which is the equilibrium of the repeated Cournot game with a marginal cost $(1-\delta)p^+$.

There are two definitions for irreversible investment. When $p^- = 0$, which means that firms cannot sell their capacities but can "throw" it, notice that the Cournot quantities (of repeated game without cost) are available when firms have important initial capacities. When $p^- = -\infty$, H is the space above the plans $q^+(.)$.

No-move space in the duopoly case when investments are irreversible:



In this case, if the initial capacities are not in H, the only symetric equilibrim is $(q^+, ..q^+)$, and firms invest at this level of capacities when $K_0 \leq (q^+, ..q^+)$. In the other case, there is an asymmetry at the equilibrium.

3 Results on collusion

Folk theorem

In this part, we find a folk theorem using some "little" restriction on our model. More precisely, we show that if we discretize the set of feasible capacities and quantities of the firms, then, when $\delta \to 1$, our model is equivalent to the infinite Cournot repeated game without any cost. First we define the discrete game considered, and present the links between the continuous game. After that, we present this Folk theorem and then discuss the usefulness of such asymptotic result to understand the effect of investment.

Definition 2 *: For $k \in \mathbb{N}$, let $\{0, A\}_k = \{0, \frac{1}{2^k}A, \frac{2}{2^k}A, .., A\}$ be a discrete approximation of $[0, A]$. Let $G_k([0, A])$ be the investment game described in part 1, when quantities and capacities are choosen in the discrete space $\{0, A\}_k$ and not in $[0, A]$. Let $E_\delta^k(K_0)$ be the set of sub-game perfect equilibrium payoffs of the investment game $G_k([0, A])$.*

Theorem 2 (Folk Theorem) *: For all $k \in \mathbb{N}$, and for all $K_0 \in_k \{0, A\}^n$, we have,*

$$\lim_{\delta \to 1} E_\delta^k(K_0) = F^*, \tag{5}$$

where F^* is the set of the feasible payoffs of the infinite Cournot repeated game. It means,

$$F^* = \left\{ \pi \in [0, A]^n : \sum_{i=1}^{n} \pi^i < \left(\frac{A}{2}\right)^2 \right\}. \tag{6}$$

Proof. : The formal proof is reported in annex 3. ùit applies the Folk Theorem of the article "Recursive Methods in Discounted Stochastic Games:

an Algorithm for $\delta \to 1$ and a Folk Theorem" of Johannes Hörner, Takuo Sugaya, Satoru Takahashi and Nicolas Vielle (2010) to our case. Intuitively, the runk assumptions represent the fact that players can identify the actions taken by the others. As our game faces no uncertainty and no imperfect monitoring, these runk conditions are verified. A first part verifies this runk condition, and a second part characterizes the individualy rational and feasible payoffs. ■

The theorem is obtained for the discrete game $G_k([0, A])$ and not for the continuous game $G([0, A])$. In fact, even if $\{0, A\}_k \to [0, A]$ when $k \to +\infty$, the formal link between the discrete and the continuous game is not easy to understand. Intuitively we should have $E_\delta(K_0) = \lim_{k \to +\infty} E_\delta^k(K_0)$ or $E_\delta(K_0) = \cap_k E_\delta^k(K_0)$., but this has not yet been proven, also the following results reinforce this initial intuition.

Consider the investment game with continous payoff (as $G([0, A])$) but with a payoff function Π_k which is a discrete approximation of Π (the payoff function of $G([0, A])$);

$$\Pi^k \left(\left(q_t^i, K_t^i\right)_{t=0,..,+\infty}^{i=1,..,n} \right) = \Pi \left(\left(\frac{\lfloor (2k+1) q_t^i \rfloor}{2k+1}, \frac{\lfloor (2k+1) K_t^i \rfloor}{2k+1} \right)_{t=0,..,+\infty}^{i=1,..,n} \right),$$

where $\lfloor . \rfloor$ is the integer part. Then this game has the same sub-game perfect equilibria than $G_k([0, A])$. Furthermore, any sub-game perfect equilibrium of $G([0, A])$ is a sub-game perfect equilibrium for Π^k. Indeed, after any history, if a deviation for one player is profitable for Π^k, it will also be profitable for Π. So $E_\delta(K_0) \subset E_\delta^k(K_0)$. It is more complicated to obtain the other inclusion, as the set of sub-game perfect equilibria in $G_k([0, A])$ is bigger than in $G([0, A])$ (less deviations are feasible). So, the following Folk theorem is shown for $G_k([0, A])$, but is not alreay extended to $G([0, A])$.

This result states that any payoff of the usual repeated Cournot game can be approximated, when δ is close enough to 1, by some payoff resulting of a sub-game perfect equilibrium of our discrete investment game $G_k([0, A])$. Note that the limit set obtained is the same for all k, so intuitively we should have the same result in the continuous case. However, this form of result is sufficient for economic interpretations, as we can restrict the investment to unit of capacity as small as we want.

We have that the limit set of equilibrium payoffs do not depend of the initial capacities, no more than the investment price p^+ or p^-. This is due to the fact that when players are patient enough, there is no cost of investing, as the price is divided in an infinity of periods which have the same weight when the players are infinitely patient ($\delta \sim 1$). The interest of such limit result is less clear than in the usual case. In the Cournot repeated game, the profit of the monopoly is $\left(\frac{A}{2}\right)^2$, and the Folk Theorem gives that there exists sub-game perfect Nash equilibria in which the industry makes profits close to the monopoly profits when firms are patient enough. However in our case the monopoly profits (and strategies) depend on δ, as shown in the following corollary.

Corollary 1 (monopoly case) *: If there is an only firm on the market, with initial capacity K_0, then its optimal strategy is to invest at the first period, to not move its capacity level after that, and to produce at full capacity:*

$$\begin{aligned} \forall t &> 0, \\ q_t &= K_t = \tilde{q}(0) = \left\{ \begin{array}{c} \frac{A-(1-\delta)p^+}{2} \ \textit{if}\ 2K_0 \leq A-(1-\delta)p^+ \\ K_0^i \ \textit{if}\ A-(1-\delta)p^+ < 2K_0 < A-(1-\delta)p^- \\ \frac{A-(1-\delta)p^-}{2} \ \textit{if}\ 2K_0 \geq A-(1-\delta)p^- \end{array} \right\}, \end{aligned} \tag{7}$$

then the monopoly profit is

$$\Pi_m^* = \left\{ \begin{array}{l} \left(\frac{A-(1-\delta)p^+}{2}\right)^2 + (1-\delta)p^+K_0 \ \textit{if}\ 2K_0 \leq A-(1-\delta)p^+ \\ (A-K_0)K_0 \ \textit{if}\ A-(1-\delta)p^+ < 2K_0 < A-(1-\delta)p^- \\ \left(\frac{A-(1-\delta)p^-}{2}\right)^2 + (1-\delta)\,p^-K_0 \ \textit{if}\ 2K_0 \geq A-(1-\delta)p^- \end{array} \right\}. \tag{8}$$

Proof. : Follows from proposition 1. ■

We find a impact of initial capacities, not only on the monopoly profit, but also on optimal capacities and quantities: there also exists a no-move area (depending on initial capacities and prices p^+ and p^-). This phenomenum disappears when δ converges to 1, as strategies and profits converge to the ones of the repeated Cournot case.

Hence, the Folk Theorem only gives us that there exists equilibria payoffs of which the sum converges to the limit of the monopoly payoff (when $\delta \to 1$). As the benchmark case tends to the equilibrium of the non-repeated Cournot game (which is the benchmark of repeated Cournot game for collusion), it shows the existence of collusive equilibria. However, several questions remain:

- Given δ, what is the impact of initial capacities on the set of equilibrium payoffs?

- Can we find a δ such that there exists a collusive equilibrium which maximizes joined profits? How do initial capacities (and capacity prices) impact it?

- For a given level of profit, what are the discounted rate allowing the existence of an equilibrium under which firms make such profit? How do initial capacities impact it?

-How can collusion be sustained? Do firms need to use punishment strategies on capacities, quantities or both?

A collusive equilibrium

In this part we study a particular case of collusion, when firms uses a Grim-Trigger strategy on capacities and we etablishes a sufficient condition to prove that this is a sub-game perfect ash equilibrium. The other kind of collusion, (using only quantities, or capacities and quantities) is not study. Furthermore, the question of how companies install collusion is not considered. In this part we will assume that firms have small initial capacities ($K_0^i < \frac{A-(1-\delta)p^+}{n+1}$).

Definition 3 *: We defined the no-move Grim-Trigger strategy as the strategy which consists for a firm to keep its initial capacity as long as the other* *firms do not move their capacity, and to go to the benchmark equilibrium when someone deviates. Formally,*

$$S_i = \left\{ \begin{array}{c} (K_t^i, q_t^i) = (K_0^i, K_0^i) \textit{ if for all } j,\ K_{t-1}^j = K_0^j \\ (K_t^i, q_t^i) = (\breve{q}^i \left[K_{t-1}\right], \breve{q}^i \left[K_{t-1}, \right]) \textit{ in the other case} \end{array} \right\}. \quad (9)$$

It is obvious that this strategy is a markovian strategy, as it only depends of the state of the industry on the previous period. As the Grim-Trigger strategy must be an equilibrium after any history, the good punishment after deviation must be the competitive equilibrium of firms depending of the capacity installed after that the deviation and not the competitive equilibrium depending of initial capacity which may not be an equilibrium after the change of capacity due to deviation. So the impact of a deviation is double, changing the capacities at a precise time, and changing the competitive equilibrium which follow. This imply that an industry which suffers a period of collusion can, at the end of this period, have a different structure (in the mean of capacities held by companies) that before the collusion. Hence collusion have long-time impact on a market, even if the collusion period is short. Note that the impact on the welfare of this change of structure is not necessarily bad. The following result give a necessary and sufficient condition for the existence of such no-move Grim-Trigger strategy in the duopoly case.

Proposition 3 (Duopoly case) *: When $n = 2$, firm i has no interest to deviate from the no-move Grim-Trigger strategy as long as the following condition is verified:*

$\cdot$ *if* $K_0^i > \frac{A-(1-\delta)(2p^+-p^-)}{3}$,

$$K_0^i(A - K_0^i - K_0^j) - \left(\frac{A - (1-\delta)p^+ - K_0^j}{2}\right)^2 - (1-\delta)p^+ K_0^i > 0, \quad (10)$$

$\cdot$ *if* $K_0^i < \frac{A-(1-\delta)(2p^+-p^-)}{3}$,

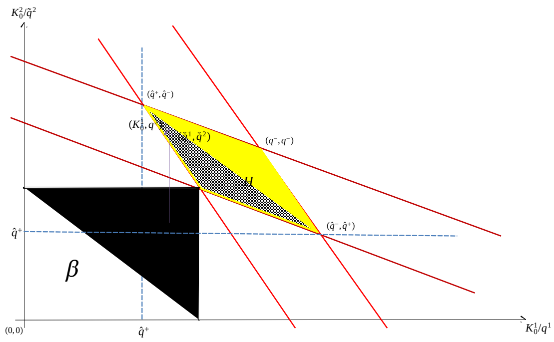
$$K_0^i(A - K_0^i - K_0^j) - \left(\frac{A - (1-\delta)p^+ + \delta p^- - K_0^j}{2}\right)^2 - (1-\delta)p^+ K_0^i + \delta(1-\delta)p^- \frac{A - (1-\delta)(2p^+ - p^-)}{3} > 0.$$

So the no-move Grim-Trigger strategy is a sub-game perfect Nash equilibrium if and only if both firms verify this condition.

Proof. : see Annex 4. Note that the one shot deviation principle permits to study the optimal deviation of the first period without lost of generality. ■

The condition (10) has some unusual properties. Indeed, increasing the discount rate decrease the possibility of collusion. So more the firms are patient, more the collusion is diffucult to sustain. This results, in opposition with the usual results on collusion is due to the fact the deviating firm deviate to a competitive equilibirum, and so there is no punishment in stricto sensu (see in the following graphic).

deviation in the first case:



So the discount factor only impact the "real" price of capacity: more patient firms can paid the capacity necessary to deviating at a higher price today, as they value more the profit they will make in the next period. Note that the selling price has no impact neither, as nobody will sell capacity. Note that this reasoning are not valuable in the second condition, it means when the initial capacity of firm is low.

In addition to these unusual results, the next proposition give a sufficient condition for the existence of the no-move Grim-Trigger strategy for a greater number of firms, and the numerical tables which follow seems to prove that the number of firms has also an unusual impact.

Proposition 4 *: The no-move Grim-Trigger strategy is a sub-game perfect equilibrium if for all i,*

$$(A-\bar{K}_0)K_0^i - \delta\left(\frac{A+n(1-\delta)p^+)}{n+1} - (1-\delta)p^-\right)\left(\frac{A+n(1-\delta)p^+)}{n+1}\right) \\ -(1-\delta)\delta p^+ K_0^i - (1-\delta)\left(\frac{A-K_0^{-i}-p^+ + \delta p^-}{2}\right)^2 > 0$$

where $\bar{K}_0 = \sum_{i=1}^n K_0^i$.

Proof. : Assume than firm i deviates at time 1 to a capacity K^i and a quantity q^i. After that, firm i must play the benchmark equilibrium. The maximal profit for a benchmark equilibrium is smaller than the maximal quantity produced when capacities belong to H multiplied by the maximal price. The maximal quantity is $\frac{A-(1-\delta)[(n+1)p^- - np^+]}{n+1}$ and the maximal price, $A - \frac{n(A-(1-\delta)p^+)}{n+1}$. So the maximal profit in H is smaller than

$$\left(\frac{A-(1-\delta)[(n+1)p^- - np^+]}{n+1}\right)\left(\frac{A+n(1-\delta)p^+)}{n+1}\right) \\ = \left(\frac{A+n(1-\delta)p^+)}{n+1} - (1-\delta)p^-\right)\left(\frac{A+n(1-\delta)p^+)}{n+1}\right).$$

Furthermore, the immediate profit for deviating is $(A-q^i-K_0^{-i}-p^+)q^i + p^+K_0^i + \delta p^- q^i$, as firms are assumed to have initial capacities smaller than the capacities in H, which is maximized in $q^i = \frac{A-K_0^{-i}-p^+ + \delta p^-}{2}$. So the maximal immediate profit for deviating is smaller than

$$(1-\delta)\left(\frac{A-K_0^{-i}-p^+ + \delta p^-}{2}\right)^2 + (1-\delta)p^+K_0^i.$$

■

The following tables gives numerical example of the condition (**??**), so positive numbers imply collusion, but the reverse is not true, as the condition is only sufficient. Note that this is obtain for symetric initial capacities which maximize the joint profit. As it is show in the corollary 1, there is several industry level of capacity which maximizes the joint profis. The first number correspond to the smallest such capacity and the second to the greatest.

$A = 15, p^+ = 2, p^- = 1:$

$\delta \backslash n$	2	3	4
0.7	$(-0.36; -0.24)$	$(0.30; 0.44)$	$(0.37; 0.51)$
0.8	$(0.43; 0.49)$	$(1.49; 1.55)$	$(1.71; 1.77)$
0.9	$(1.60; 1.61)$	$(2.95; 2.97)$	$(3.28; 3.29)$

$A = 15, p^+ = 3, p^- = 1:$

$\delta \backslash n$	2	3	4
0.7	$(-1.28; -1.08)$	$(-0.37; -0.14)$	$(-0.12; 0.12)$
0.8	$(-0.34; -0.26)$	$(0.92; 1.03)$	$(1.31; 1.41)$
0.9	$(1.13; 1.15)$	$(2.62; 2.64)$	$(3.04; 3.06)$

$A = 15, p^+ = 3, p^- = 2:$

$\delta \backslash n$	2	3	4
0.7	$(-1.11; -0.99)$	$(-0.32; -0.18)$	$(-0.16; -0.02)$
0.8	$(-0.25; -0.20)$	$(0.92; 0.99)$	$(1.24; 1.30)$
0.9	$(1.16; 1.17)$	$(2.60; 2.61)$	$(2.98; 3.00)$

$A = 5, p^+ = 3, p^- = 2:$

$\delta \backslash n$	2	3	4
0.7	$(-1.11; -0.99)$	$(-0.32; -0.18)$	$(-0.16; -0.02)$
0.8	$(-0.25; -0.20)$	$(0.92; 0.99)$	$(1.24; 1.30)$
0.9	$(1.16; 1.17)$	$(2.60; 2.61)$	$(2.98; 3.00)$

This tables seem to indicate that an increase of the number of firms increase the possibility of collusion. This can be due to the fact, when the number of firms increase, punishment is more severe and deviation more expensive (even if it is more profitable after the choice of capacity).

This part shows that there exists collusive equilibrium using no-move Grim-Trigger strategy, which implies that firms produce at full capacity as in the competitive equilibrium, and that their capacities remain constant in time. Furthermore, the condition of the existance of such equilibrium can be surprising due to the usual condition of collusion. Detecting this kind of collusion may be difficult, as the only difference between competitive equilibrium is in level of capacity.

4 Annex

Annex 1: Best responses in the non-collusive equilibrium

Proof. [Proof (Best response)] : Let $q^{i+} = \tilde{q}^{+}(q^{-i})$ and $q^{i-} = \tilde{q}^{-}(q^{-i})$

Assume that the strategy of firm i does not depend of the time or of the game's history, so $(K_t^i, q_t^i) = (q^i, q^i)$. Then the objective of player is to maximize:

$$\Pi^i(q^i, q^i, q^{-i}, q^{-i}) = (A - q^i + q^{-i})q^i - (1-\delta)p^{+}(q^i - K_0^i)^{+} + (1-\delta)p^{-}(q^i - K_0^i)^{-}, \tag{11}$$

which can be rewritten as:

$$\Pi^i(q^i, q^i, q^{-i}, q^{-i}) = \left\{ \begin{array}{c} (A - q^i + q^{-i})q^i + (1-\delta)p^{-}(K_0^i - q^i) \text{ if } q^i < K_0^i \\ (A - q^i + q^{-i})q^i \text{ if } q^i = K_0^i \\ (A - q^i + q^{-i})q^i - (1-\delta)p^{+}(q^i - K_0^i) \text{ if } q^i > K_0^i \end{array} \right\}.$$

This function is obviously continous in q^i.

If $q^{i+} \leq q^{i-} < K_0^i$, the function is increasing until q^{i-}, then decreasing until K_0^i, and then decreasing after that, So the function is maximum in q^{i-}.

If $K_0^i < q^{i+} \leq q^{i-}$, the function is increasing until K_0^i, then increasing until q^{i+}, and then decreasing after that, so the function is maximum in q^{i+}.

If $q^{i+} \leq K_0^i \leq q^{i-}$, the function is increasing until $\min(q^{i-}, K_0^i) = K_0^i$, then decreasing after $\max(K_0^i, q^{i+}) = K_0^i$. So the function is maximum in K_0^i.

We have

$$q^{*i} = \left\{ \begin{array}{c} q^{i+} \text{ if } q^{i-} < K_0^i \\ K_0^i \text{ if } q^{i+} \leq K_0^i \leq q^{i-} \\ q^{i-} \text{ if } K_0^i < q^{i+} \end{array} \right\}. \tag{12}$$

Now we will see that the firm cannot do better by using a more complex strategy (remind that we assume here that other firms play a static strategy).

This is a dynamic programing issue, with control on q_t^i and with objective function:

$$\begin{aligned} \Pi^i(q_t^i, q_t^i, q^{-i}, q^{-i}) &= (1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1} \left[(A - q_t^i + q^{-i})q_t^i - p^+(q_t^i - q_{t-1}^i)^+ + p^-(q_t^i - q_{t-1}^i)^-\right] \\ \Pi^i(q_t^i, q_t^i, q^{-i}, q^{-i}) &= (1-\delta)\left[(A - q_1^i + q^{-i})q_1^i - p^+(q_1^i - K_0^i)^+ + p^-(q_1^i - K_0^i)^-\right] \\ &\quad +\delta\Pi^i(q_{t+1}^i, q_{t+1}^i, q^{-i}, q^{-i}) \end{aligned}$$

Let $\Pi^{*i}(K_0^i)$ be the the maximum of profit the firm can make with initial capacity K_0^i. Than Π^{*i} verifies:

$$\Pi^{*i}(K_0^i) = \max_{q_1^i} \left[v(q_1^i, K_0^i) + \delta\Pi^{*i}(q_1^i)\right], \tag{13}$$

where

$$v(q_1^i, K_0^i) = (1-\delta)\left[(A - q_1^i + q^{-i})q_1^i - p^+(q_1^i - K_0^i)^+ + p^-(q_1^i - K_0^i)^-\right].$$

If $q_1^i \geq q^{*i}$, then

$$\frac{v(q_1^i, K_0^i)}{(1-\delta)} = (A - q_1^i + q^{-i})q_1^i - p^+(q^{*i} - K_0^i)^+ + p^-(q^{*i} - K_0^i)^- - p^+(q_1^i - q^{*i}),$$

(because $(.)^-$ and $(.)^+$ are linear). Furthermore,

$$\Pi^{*i}(q^{*i}) \geq \Pi^{*i}(q_1^i) - (1-\delta)p^+(q_1^i - q^{*i}).$$

This is the case because there exists an optimal solution (the theorem of Tychonoff gives us that (in french, l'ensemble des suites à valeurs dans un compact est compact) the set of sequences with values in a compact is compact, and $\Pi^i(q_t^i, q_t^i, q^{-i}, q^{-i})$ is continuous in q_t^i); so from q^{*i} we can go to the optimal solution from q_1^i at a cost $p^+(q_1^i - q^{*i})$. This implies:

$$v(q_1^i, K_0^i) + \delta\Pi^{*i}(q_1^i) \leq v(q_1^i, K_0^i) + \delta\Pi^{*i}(q^{*i}) + \delta(1-\delta)p^+(q_1^i - q^{*i}).$$

So, for $q_1^i > q^{*i}$, maximizing equation (13) is equivalent to maximize:

$$(A - q_1^i + q^{-i})q_1^i - p^+(q_1^i - q^{*i}) - \delta p^+(q_1^i - q^{*i}).$$

We know that

$(A-q_1^i+q^{-i})q_1^i-(1-\delta)p^+(q_1^i-q^{*i})$ is maximized in $q_1^i = \dfrac{A - q^{-i} - (1-\delta)p^+}{2}$,

and as $\frac{A-q^{-i}-(1-\delta)p^+}{2} \leq q^{*i}$, $v(q_1^i, K_0^i) + \delta\Pi^{*i}(q_1^i)$ is increasing until q^{*i}.

By the same reasoning, we have, if $q_1^i \leq q^{*i}$:

$$\frac{v(q_1^i, K_0^i)}{(1-\delta)} = (A - q_1^i + q^{-i})q_1^i - p^+(q^{*i} - K_0^i)^+ + p^-(q^{*i} - K_0^i)^- + p^-(q^{*i} - q_1^i),$$

and

$$\Pi^{*i}(q_1^i) \geq \Pi^{*i}(q^{*i}) + (1-\delta)p^-(q^{*i} - q_1^i).$$

This implies the fact that maximizing equation (13) is equivalent to maximize:

$$(A - q_1^i + q^{-i})q_1^i + p^-(q^{*i} - q_1^i) + \delta p^-(q^{*i} - q_1^i),$$

and so $v(q_1^i, K_0^i) + \delta\Pi^{*i}(q_1^i)$ is decreasing until q^{*i}.

$q^{*i}(K_0^i)$ maximizes (13), and, as $q^{*i}(q^{*i}(K_0^i)) = q^{*i}(K_0^i)$, the optimal solution is $q_t^i = q^{*i}(K_0^i)$. ■

Annex 2: The benchmark equilibrium

Proof. [Proof (Benchmark equilibrium)] : We start to prove that, if $\hat{q} = (\hat{q}^1, .., \hat{q}^n)$ is a sub-game perfect equilibrium of the benchmark form, it minimizes $\sum_{i=1}^n |q^i - K_0^i|$ for $q = (q^1, .., q^n) \in H$ (1). Then, we prove there exists only one $q \in H$ which minimizes $\sum_{i=1}^{n} |q^i - K_0^i|$ (2). After that we prove that if $\hat{q}$ minimizes $\sum_{i=1}^{n} |q^i - K_0^i|$ for $q \in H$, it is a sub-game perfect equilibrium (3). This etablishes the existence and unicity of the sub-game perfect equilibrium and characterizes it as the $\arg\min_{(q^1,..,q^n)\in H} \sum_{i=1}^{n} |q^i - K_0^i|$.

(1) Assume that $\hat{q} = (\hat{q}^1, .., \hat{q}^n)$ is a sub-game perfect equilibrium. Let

$$\begin{aligned} E^+ &= \{i \in \{1, .., n\} : \hat{q}^i = \tilde{q}^+(\hat{q}^{-i})\} = \left\{i \in \{1, .., n\} : K_0^i < \hat{q}^i\right\} \\ E^- &= \{i \in \{1, .., n\} : \hat{q}^i = \tilde{q}^-(\hat{q}^{-i})\} = \left\{i \in \{1, .., n\} : K_0^i > \hat{q}^i\right\} \\ E^0 &= \{i \in \{1, .., n\} : \hat{q}^i = K_0^i\} = \left\{i \in \{1, .., n\} : \tilde{q}^-(\hat{q}^{-i}) \le K_0^i \le \tilde{q}^+(\hat{q}^{-i})\right\}. \end{aligned}$$

For $q \in H$, let

$$I_q = \{i \in \{1, .., n\} : q^i \neq \hat{q}^i\}.$$

Let P_k be the proposition: for all $q \in H$, such that $card(I_q) = k$, then $d(q, K_0) \ge d(\hat{q}, K_0)$. We will show it by recurrence.

For $k = 1$, $q = \hat{q}$ except in one i. There are three possibilities:

$\cdot$ $i \in E^0$, then $\hat{q}^i = K_0^i$ and $|\hat{q}^i - K_0^i| \le |q^i - K_0^i|$.

$\cdot$ $i \in E^+$, then as $q \in H$, $q^i \ge \tilde{q}^+(q^{-i}) = \tilde{q}^+(\hat{q}^{-i}) = \hat{q}^i > K_0^i$, so $q^i - K_0^i \ge \hat{q}^i - K_0^i > 0$.

$\cdot$ $i \in E^-$, then as $q \in H$, $q^i \le \tilde{q}^-(q^{-i}) = \tilde{q}^-(\hat{q}^{-i}) = \hat{q}^i < K_0^i$, so $q^i - K_0^i \le \hat{q}^i - K_0^i < 0$.

This implies that $|q^i - K_0^i| \ge |\hat{q}^i - K_0^i|$ so $\sum_{i=1}^{n} |q^i - K_0^i| \ge \sum_{i=1}^{n} |\hat{q}^i - K_0^i|$.

Assume that P_{k-1} is true.

Let $q \in H$, such that $card(I_q) = k$. Then let $\bar{\imath} = \arg\max_{i\in I_q}(|q^i - K_0^i| - |\hat{q}^i - K_0^i|)$. Let $\bar{q}^j = q^j$ if $j \neq \bar{\imath}$ and $\bar{q}^{\bar{\imath}} = \hat{q}^{\bar{\imath}}$. We have that $card(I_{\bar{q}}) = k - 1$, so, by P_{k-1}, $d(\bar{q}, K_0) \ge d(\hat{q}, K_0)$, which means

$$\sum_{I_q \backslash \bar{\imath}} (|q^i - K_0^i| - |\hat{q}^i - K_0^i|) \ge 0.$$

But $(|q^{\bar{\imath}} - K_0^{\bar{\imath}}| - |\hat{q}^{\bar{\imath}} - K_0^{\bar{\imath}}|) \ge (|q^i - K_0^i| - |\hat{q}^i - K_0^i|)$ for all $i \in I_q \backslash \bar{\imath}$. So $(|q^{\bar{\imath}} - K_0^{\bar{\imath}}| - |\hat{q}^{\bar{\imath}} - K_0^{\bar{\imath}}|) \ge 0$. Hence, we have:

$$\sum_{I_q} (|q^i - K_0^i| - |\hat{q}^i - K_0^i|) \ge 0$$

$$\Leftrightarrow d(q, K_0) \ge d(\hat{q}, K_0).$$

By recurrence, P_k is true for all $k = 1,..n$, so if $\hat{q}$ is an equilibrium, $\hat{q} \in \arg\min_{q\in H} \sum_{i=1}^{n} |q^i - K_0^i|$.

(2) Now, as by proposition 1, $H \neq \varnothing$, there exists at least one $q \in H$ which minimizes $\sum_{i=1}^{n} |q^i - K_0^i|$, we have to show that this q is unique. For that we assume that there exists $\hat{q}$ and $\check{q}$ in $\arg\min_{q\in H} \sum_{i=1}^{n} |q^i - K_0^i|$ such that $\hat{q} \neq \check{q}$.

In this case we have $[\check{q}, \hat{q}] \subset \arg\min_{q\in H} \sum_{i=1}^{n} |q^i - K_0^i|$.

(Indeed, $\forall q \in [\check{q}, \hat{q}]$, $\sum_{i=1}^{n} |q^i - K_0^i| = \lambda \sum_{i=1}^{n} |\check{q}^i - K_0^i| + (1-\lambda) \sum_{i=1}^{n} |\hat{q}^i - K_0^i|$, so $\sum_{i=1}^{n} |q^i - K_0^i| = \min_{q\in H} \sum_{i=1}^{n} |q^i - K_0^i|$).

Furthermore, if $q \in \arg\min_{q\in H} \sum_{i=1}^{n} |q^i - K_0^i|$ and $K_0 \notin H$, $q \in fr(H)$.

(Indeed, if $q \notin fr(H)$, $\exists \bar{q} \in [q, K_0^i]$ such that $\bar{q} \in H$, i.e. $\exists \lambda \in]0,1[$ such that $\bar{q} = \lambda q + (1-\lambda) K_0^i \in H$. Hence, $\sum_{i=1}^{n} |\bar{q}^i - K_0^i| = \lambda \sum_{i=1}^{n} |q^i - K_0^i| < \min_{q\in H} \sum_{i=1}^{n} |q^i - K_0^i|$, which is a contradiction).

Note that if $K_0 \in H$, there is a single obvious solution, $q = K_0$.

Then we show that $\exists i \in \{1,..,n\}, s \in \{+,-\}$ such that $\forall q \in [\check{q}, \hat{q}]$, $q^i = \tilde{q}^s(q^{-i})$.

Assume that there is no $s \in \{+,-\}, i \in \{1,..,n\}$ such that $\hat{q}^i = \tilde{q}^s(\hat{q}^{-i})$ and $\check{q}^i = \tilde{q}^s(\check{q}^{-i})$. This means that for all $s \in \{+,-\}$,

$$i \in \{1,..,n\}, \hat{q}^i \neq \tilde{q}^s(\hat{q}^{-i}) \text{ and } \check{q}^i \neq \tilde{q}^s(\check{q}^{-i}).$$

In this case let $\bar{q} = \frac{1}{2}\check{q} + \frac{1}{2}\hat{q}$, then for all i, s,

$$\bar{q}^i \neq \frac{1}{2}\tilde{q}^s(\check{q}^{-i}) + \frac{1}{2}\tilde{q}^s(\hat{q}^{-i}) = \tilde{q}^s(\bar{q}^{-i}),$$

which contradicts the fact that $\bar{q} \in fr(H)$. So there exists $i \in \{1,..,n\}, s \in \{+,-\}$, such that $\hat{q}^i = \tilde{q}^s(\hat{q}^{-i})$ and $\check{q}^i = \tilde{q}^s(\check{q}^{-i})$, hence

$$\forall \lambda, \text{ if } q_\lambda = \lambda\check{q} + (1-\lambda)\hat{q}, \text{ then } q_\lambda^i = \tilde{q}^s(q_\lambda^{-i}).$$

Now, we can contradict the assumption $\hat{q} \neq \check{q}$.

Let, $\lambda \in]0,1[$, then $\exists \hat{\varepsilon}$ such that, for $|\varepsilon| \leq \hat{\varepsilon}$, $q_{\lambda+\varepsilon} \in [\check{q}, \hat{q}]$, so, for some s, i $q^i_{\lambda+\varepsilon} = \tilde{q}^s(q^{-i}_{\lambda+\varepsilon}) = \tilde{q}^s(q^{-i}_{\lambda}) - \frac{1}{2}(\varepsilon\check{q} - \varepsilon\hat{q})^{-i}$, so $q^i_{\lambda+\varepsilon} = q^i_\lambda - \frac{\varepsilon}{2}(\check{q} - \hat{q})^{-i} = q^i_{\lambda+\varepsilon} - \varepsilon(\check{q} - \hat{q})^i - \frac{\varepsilon}{2}(\check{q} - \hat{q})^{-i}$. Let $\gamma = \frac{1}{2}\left[(\check{q} - \hat{q})^i + \sum_{i=1}^{n}(\check{q} - \hat{q})^i\right]$, then

$\sum_{j=1}^{n} \left| q_{\lambda+\varepsilon}^{j} - K_0^j \right| = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| q_{\lambda+\varepsilon}^{j} - K_0^j \right| + \left| q_{\lambda+\varepsilon}^{i} - K_0^i - \varepsilon\gamma \right|$. If $\varepsilon = \frac{q_{\lambda+\varepsilon}^{i} - K_0^i}{\left| q_{\lambda+\varepsilon}^{i} - K_0^i \right| \gamma} \max(\hat{\varepsilon}, 1)$, then $|\varepsilon| \leq \hat{\varepsilon}$ and $\left| q_{\lambda+\varepsilon}^{i} - K_0^i - \varepsilon\gamma \right| = \left| q_{\lambda+\varepsilon}^{i} - K_0^i \right| - |\varepsilon\gamma|$, so $\sum_{j=1}^{n} \left| q_{\lambda+\varepsilon}^{j} - K_0^j \right| = \sum_{j=1}^{n} \left| q_{\lambda+\varepsilon}^{j} - K_0^j \right| - |\varepsilon\gamma|$, as $\gamma > 0, \varepsilon > 0$, this not possible.

So, there is only one $q \in H$ which minimizes $\sum_{i=1}^{n} \left| q^i - K_0^i \right|$.

(3) Assume $\hat{q} = \min_{q \in H} \sum_{i=1}^{n} |q^i - K_0^i|$. Fixe $i \in \{1, .., n\}$. Assume that $\check{q}$ is not an equilibrium. Then there exists i such that $\tilde{q}(\check{q}^{-i}) \neq \check{q}^i$.

To prove that $\check{q}$ is an equilibrium it is sufficient to see that $\tilde{q}(\check{q}^{-i}) = \check{q}$. We know that $\check{q} \in H$. Then, let $\bar{q}^j = \check{q}^j$ if $j \neq i$ and $\bar{q}^i = \tilde{q}(\check{q}^{-i})$, we have $\bar{q} \in H$. Then there are three possibilities:

- $\tilde{q}(\check{q}^{-i}) = K_0^i$, then

$$\sum_{i=1}^{n} \left| \bar{q}^i - K_0^i \right| \leq \sum_{i=1}^{n} \left| \check{q}^i - K_0^i \right|,$$

so $\bar{q} = \check{q}$ and $\bar{q}^i = \tilde{q}(\check{q}^{-i})$.

- $\tilde{q}(\check{q}^{-i}) > K_0^i$, then $\bar{q}^i = \tilde{q}^{+}(\check{q}^{-i}) > K_0^i$, so in this case $\check{q}^i \geq \tilde{q}^{+}(\check{q}^{-i})$ (as if $q \in H$, $q^i \geq q^{+}(q^{-i})$) and $\check{q}^i \geq K_0^i$, so

$$\sum_{i=1}^{n} \left| \bar{q}^i - K_0^i \right| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \bar{q}^j - K_0^j \right| + \bar{q}^i - K_0^i \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \check{q}^j - K_0^j \right| + \check{q}^i - K_0^i \leq \sum_{i=1}^{n} \left| \check{q}^i - K_0^i \right|,$$

so $\bar{q} = \check{q}$ and $\bar{q}^i = \tilde{q}(\check{q}^{-i})$.

- $\tilde{q}(\check{q}^{-i}) < K_0^i$, then $\bar{q}^i = \tilde{q}^{-}(\check{q}^{-i}) < K_0^i$, so in this case $\check{q}^i \leq \tilde{q}(\check{q}^{-i})$ (as if $q \in H$, $q^i \leq q^{-}(q^{-i})$) and $\check{q}^i \leq K_0^i$, so

$$\sum_{i=1}^{n} \left| \bar{q}^i - K_0^i \right| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \bar{q}^j - K_0^j \right| + K_0^i - \bar{q}^i \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \check{q}^j - K_0^j \right| + K_0^i - \check{q}^i \leq \sum_{i=1}^{n} \left| \check{q}^i - K_0^i \right|,$$

so $\bar{q} = \check{q}$ and $\bar{q}^i = \tilde{q}(\check{q}^{-i})$. ■

Annex 3: The Folk Theorem

Proof. : *Part 1:* We use the theorem 2 (page 23) of the article of Johannes Hörner and all (2010), with $Y =_k [0, A]^n$ is the set of quantities played by the players at the previous round, $A^i =_k \{0, A\}^2$ is the set of (K^i, q^i) chosen at one period, and $S =_k \{0, A\}^n$ is the set of capacities at the previous time. Let $E^k_\delta(K_0)$ be the set of all sub-game perfect equilibrium payoffs in $GI_k([0, A])$.

Let $R^i\,[K_0, (K^j, q^j)_{j\neq i}]$ be the matrix of size ${}_k\{0, A\}^2 \times_k \{0, A\}^n$ such that, for all $(K^i, q^i), K_1, q_1$

$$R^i\left[(K^i, q^i); K_1, q_1\right] = 1_{\{K_1^j = K^j \text{ and } q_1^j = q^j, \forall j=1,..n\}},$$

so R^i has rank $card\,({}_k\{0, A\}^2)$. So every pure action has individual full rank for all players.

Let

$$R^{ik}\left[K_0, (K^j, q^j)_{j=1}^n\right] = \begin{pmatrix} R^i\,[K_0, (K^j, q^j)_{j\neq i}] \\ R^k\,[K_0, (K^j, q^j)_{j\neq k}] \end{pmatrix}.$$

Note that $R^i\,[(K^i, q^i); K_1, q_1] = 1 = R^i\,[(K^j, q^j); K_1, q_1]$ if and only if $K^i = K_1^i$, $K^j = K_1^i$, $q^i = q_1^i$ and $q^j = q_1^i$ which arrises only one time on ${}_k\{0, A\}^2 \times_k \{0, A\}^2$. This implies that R^{ik} has rank $2 * card\,({}_k\{0, A\}^2) - 1$, so for each state K_0 and each pair (i, k) of players, all pure strategies have pairwise full rank.

So conditions $F1$ and $F2$ of the article of Hörner and all (2010) are verified, and we can use the theorem.

Part 2 : We find the set of feasible and individualy rational payoffs of the stochastic game.

Let $m^i_\delta(K_0)$ be the min-max value of player i (in the game $GI_k([0, A]^n)$). We show that $m^i_\delta(K_0) \leq (1-\delta)p^- K_0^i$.

Indeed, if the other players than i play the strategy $q_t^{-i} = K_t^{-i} = A$ then the best response of player i is to sell all his capacity as soon as possible. In this case he makes a profit of $(1-\delta)p^- K_0^i$. So $m^i_\delta(K_0) \leq (1-\delta)p^- K_0^i$. Hence $\lim_{\delta\to 1} m^i_\delta(K_0) = 0$. The Folk theorem of Hörner and all (2010) gives that the limit set of the feasible payoff of the game $GI_k([0, A]^n)$ is the limit of $E^k_\delta(K_0)$ when $\delta \to 1$.

Now, we will show that the limit set of all the feasible payoffs in $GI_k([0, A]^n)$ is F^*.

First, we will see that, if we call $F^k_\delta(K_0)$ the set of feasible payoffs in $GI_k([0,A]^n)$, $F^* \subset \lim_{\delta\to 1} F^k_\delta(K_0)$.

By using the strategy $(0, K^i_0)$ the player i can have 0 profit, so $(0,..,0)$ is feasible for all $F^k_\delta(K_0)$.

In the same idea if the player i plays the monopoly strategy $\left(\frac{A}{2}, \frac{A}{2}\right)$ and all other firms plays the previous strategy, i has a profit of

$$\left(\frac{A}{2}\right)^2 - (1-\delta)p^+\left(\frac{A}{2} - K^0_i\right)_+ + (1-\delta)p^-\left(\frac{A}{2} - K^0_i\right)_-,$$

and the other 0. Hence the payoffs vector which gives zero for all firm except for one and $\left(\frac{A}{2}\right)^2$ for this one, is in the limit set of feasible payoffs (as the limit of payoffs in $F^k_\delta(K_0)$). So, by convexity, $F^* \subset \lim_{\delta\to 1} F^k_\delta(K_0)$.

Now, we will show that $\cap_{K_0} \lim_{\delta\to 1} F^k_\delta(K_0) \subset F^*$. Assume that $K_0 = 0$. Then the sum of the profits of all firms can be written as

$$\sum_{i=1}^n \Pi^i = (1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}\left(\sum_{i=1}^n \pi^i_t\right)$$

$$\begin{aligned}\sum_{i=1}^n \Pi^i &= (1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}\left(\sum_{i=1}^n \tilde{\pi}^i_t\right) \\ &\quad +(1-\delta)\sum_{t=1}^{+\infty} \delta^{t-1}\left(\sum_{i=1}^n p^-(K^i_t - K^i_{t-1})^- - p^+(K^i_t - K^i_{t-1})^+\right).\end{aligned}$$

As $K_0 = 0$,

$$\sum_{t=1}^{+\infty} \delta^{t-1}\left(\sum_{i=1}^n p^-(K^i_t - K^i_{t-1})^- - p^+(K^i_t - K^i_{t-1})^+\right) \leq 0.$$

Furthermore,

$$\left(\sum_{i=1}^n \tilde{\pi}^i_t\right) = \sum_{i=1}^n (A - Q_t)q^i_t = (A - Q_t)Q_t \leq \left(\frac{A}{2}\right)^2.$$

So, $\sum_{i=1}^n \Pi^i \leq \left(\frac{A}{2}\right)^2$ and $F^k_\delta(0) \subset F^*$ for all δ. This implies that $\cap_{K_0} \lim_{\delta\to 1} F^k_\delta(K_0) = F^*$.

So, we have that

$$F^* = \lim_{\delta \to 1} E_\delta^k(K_0). \tag{14}$$

■

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